JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 16, 216-218 (1966)

# Lower Bounds for Characteristic Values*

R. M. REDHEFFER

*Department of Mathematics, University of California, Los Angeles, California*

*Submitted by R. Bellman*

It is the purpose of this note to give lower bounds for the characteristic value, $\lambda$, in a certain differential inequality. The novelty is that the differential operator is not formally self-adjoint, the conditions on the coefficients are very weak, and corners are allowed in the comparison function, $v$. The method generalizes to problems involving degenerate elliptic operators in $n$ dimensions, though details are not given here.

Throughout this note $(a, b)$ denotes a finite or infinite open interval of the real line, the independent variable is $x$, and $f, g, h, u, v, y$ are functions of $x$, with $h \geqslant 0$. For a class of functions $u$ to be specified later an operator $T$ is defined by

$$Tu = f - g\frac{u'}{u} - h\frac{u''}{u}. \tag{1}$$

This operator is associated with the characteristic-value problem

$$hu'' + gu' + (\lambda - f)\,u \geqslant 0, \tag{2}$$

where $\lambda$ is constant.

We say that a condition holds "mod $E$" (for *enumerable*) if it holds in some specified set, except perhaps at the points of a countable subset. A function $u$ is *admissible* if $u$ is continuous in $(a, b)$, $u''$ exists in $(a, b)$ mod $E$, and, in the countable subset where $u'$ might not exist, we have $u'(x\,-) \geqslant u'(x\,+)$. The latter condition could be weakened by use of Dini derivates, though this is not done here. However, the condition cannot be dropped, as is shown by consideration of the simple case $y'' + y = 0$.

We say that $u$ is a *nontrivial solution* of the inequality $Tu \leqslant \lambda$ if $u$ is admissible, $u > 0$ at some point, and $Tu \leqslant \lambda$ holds mod $E$ at all points of $(a, b)$ where $u > 0$. In these circumstances we also say that $u$ is a nontrivial solution of (2). This convention allows (2) to fail at points where $u \leqslant 0$, but as a result our assertions are stronger than they would be if we required (2) to hold everywhere.

* This work is supported in part by the Office of Naval Research Nonr-1841-38.

A *comparison function* is a positive function $v$ such that $-v$ is admissible and such that the function $w = u/v$ satisfies

$$\sup w(x) > \inf_{x \to a+} w(x), \qquad \sup w(x) > \inf_{x \to b-} w(x). \tag{3}$$

Equation (3) is assured by suitable boundary conditions. For example the three conditions

$$\liminf_{x \to a+} u(x) < 0, \qquad \liminf_{x \to a+} u(x) \leqslant 0, \qquad \liminf_{x \to a+} u(x) < \infty$$

would ordinarily entail the following conditions for $v$:

$$\text{no condition}, \qquad \liminf_{x \to a+} v > 0, \qquad \liminf_{x \to a+} u(x) = \infty.$$

Similar pairs of conditions can be given at $b$, and others can be stated which involve the derivatives rather than the values. For example, one could have

$$\liminf_{x \to b-} u'(x) \leqslant 0, \qquad \liminf_{x \to b-} v'(x) > 0.$$

In every case the purpose is to rule out a positive end-point maximum for $u/v$ as *sole* maximum.

The *set of regularity*, $R$, is the set of points of $(a, b)$ at which $u > 0$, $u''$ exists, $v''$ exists, and the differential inequality $Tu \leqslant \lambda$ actually holds. Points where $u > 0$ but one of the other conditions fails are called *singularities*. Our basic assumption is that the set of singularities is countable.

We shall establish:

REMARK 1. *With $T$ as described, let $u$ be a nontrivial solution of $Tu \leqslant \lambda$ and let $v$ be a comparison function. Suppose there exists a differentiable function $y$ such that all of its zeros are in $R$ and such that $y^2h$ and $y(gv + 2hv') + y'hv$ are bounded above in the intersection of $R$ with any compact subset of $(a, b)$. Then*

$$\lambda \geqslant \inf_{x \in R} Tv.$$

Since the one-sided bound for $y(gv + 2hv') + y'hv$ is required only in compact subintervals, and not for $x \to a+$ or $x \to b-$, it is verified automatically in practically every case of interest. However we can allow $gv + 2hv'$ to take alternating values $+\infty$ and $-\infty$, provided the alternations are separated by open intervals on which $|gv + 2hv'|$ is bounded. This is true because changes of sign of $y$ can occur in these intervals without making $y = 0$ at a singularity.

For proof define $U = \log u$ in the set where $u > 0$, and $V = \log v$. The hypothesis shows that there is a compact subinterval $[\alpha, \beta]$ of $(a, b)$ in which the function $U - V$ assumes a maximum, and the maximum actually exceeds the boundary values. Hence the same applies to

$$U - V - \epsilon \int y(x)\, dx$$

when $|\epsilon|$ is small. At the maximum it is easily seen that $u'$ and $v'$ exist, since $u$ and $-v$ are admissible, and hence

$$U' - V' = \epsilon y.$$

Since the set of singularities is countable, while the set of possible choices of $\epsilon$ is not, we can choose $\epsilon$ so that the maximum occurs in $R$. Thus,

$$U'' - V'' \leqslant \epsilon y'.$$

At this point the hypothesis $Tu \leqslant \lambda$ gives

$$\lambda \geqslant f - gU' - h[U'' + (U')^2] \geqslant Tv - \epsilon y(g + 2hV') - \epsilon h(y' + \epsilon y^2)$$

and the conclusion follows when $\epsilon \to 0$.

Conditions for strict inequality are obtained similarly. For example, the choice $y = e^{kx}$ shows that $\lambda > \inf Tv$ if

$$g < 0 \quad \text{when} \quad h = 0 \mod E, \qquad \text{and} \qquad \sup \left( \frac{g}{h} + 2 \frac{v'}{v} \right) < \infty,$$

where the sup is over the points of $R$ in $[\alpha, \beta]$ where $h(x) > 0$. Another condition can be given with inequalities reversed. Since $g < 0$ is needed only mod $E$, this requirement can be dropped if the set of zeros of $h$ is countable. Thus we obtain:

REMARK 2. *Let $u$ be a nontrivial solution of $Tu \leqslant \lambda$ and let $v$ be a comparison function. Suppose $h > 0$* mod $E$ *and suppose*

$$\frac{g}{h} + 2 \frac{v'}{v}$$

*is bounded from one side* mod $E$ *in each compact subinterval of $(a, b)$. Then*

$$\lambda > \inf_{x \in R} Tv.$$